Introduction to Mathematical Quantum Theory

Text of the Exercises

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Exercise 1

Let V be a closed subspace of $\mathcal{H}$ Hilbert space. Let A be a linear bounded operator on $\mathcal{H}$ such that $A(V) \subseteq V$. Prove that $A^*\left(V^\perp\right) \subseteq V^\perp$.

Exercise 2

Let $\mathcal{H}$ be an Hilbert space. Let A be a linear bounded operator on $\mathcal{H}$ with linear bounded inverse A^{-1}. Prove that $\left(A^{-1}\right)^* A^* = A^*\left(A^{-1}\right)^* = \mathrm{id}$. Deduce that A^* is invertible and that $(A^*)^{-1} = \left(A^{-1}\right)^*$.

Exercise 3

Consider the Hilbert space $\mathcal{H} := \ell^2(\mathbb{N})$.

a Define the operator A as

$$(A\alpha)_n = \alpha_{n+1} \qquad \forall n \in \mathbb{N}, \tag{1}$$

for any $\alpha = \{\alpha_n\}_{n\in\mathbb{N}} \in \mathcal{H}$.

Prove that A is a well defined linear bounded operator, find its norm and its spectrum.

b Consider A^* the adjoint of A. Show its explicit action and find its norm and its spectrum.

c Define $B := A^*A$. Prove that B is a self-adjoint operator, show its explicit action and find its norm and its spectrum.

Hint: Recall that if T is a linear bounded operator, the spectrum $\sigma(T)$ is a closed set, $\rho(T) \equiv \mathbb{C}\setminus\sigma(T)$ the resolvent of T is defined as

$$\rho(T) := \left\{\lambda \in \mathbb{C} \middle|\ (T - \lambda\,\mathrm{id})^{-1} \text{ is a well-defined, linear, bounded operator}\right\}, \tag{2}$$

and that $\sigma(T) \subseteq \overline{B_{\|T\|}(0)}$, where $B_R(0) := \{\alpha \in \mathcal{H} |\ \|\alpha\|_2 < R\}$.

Exercise 4

Consider the interval $I = (a, b) \subseteq \mathbb{R}$ and the Hilbert space $\mathcal{H} := L^2(I)$. Consider $\varphi \in C(I)$ a real valued continuous function with $\|\varphi\|_\infty < +\infty$. Consider the operator T_φ defined for any $\psi \in \mathcal{H}$ as

$$T_\varphi \psi(x) := \varphi(x)\psi(x). \tag{3}$$

Prove that T_φ is a well defined linear bounded operator and prove that $\sigma(T_\varphi) = \overline{\varphi(I)}$.

Hint: Show first that $\varphi(I) \subseteq \sigma(T_\varphi)$ and use the fact that the spectrum is closed to show that the same is true for the closures. Next, show that $\left(\overline{\sigma(T_\varphi)}\right)^{\mathrm{c}} \subseteq \rho(T_\varphi)$ to conclude.